The Cube and its Symmetries

This workshop is designed to show Y8 students how symmetry operations combine in 3 dimensions. We take the cube as our example. The children should be encouraged to bring cameras (or mobiles with a photographic facility) to the session. To enable you to reproduce them, I have given each of the 18 acetates its own page.

Administration

In your area is an OHP and screen and the single models you need to demonstrate.
The acetates are reproduced as masters for printing at the relevant points in these notes.
The children work in pairs. At tables accessible to the pupils are pots of interlocking cubes (Multilink) and interlocking panels (Polydron Frameworks) and the prepared models each pair needs.

There is enough material here for 2 sessions, i.e. 5 hours of work. If your aim is not to present the ***combination*** *of operations and the* ***group structure*** *encapsulating this but simply to enable the children to find how these operations manifest themselves in 3 dimensions, use the short version sketched on the last page (48).*

materials needed *activity (Teacher demonstration/Pupil experiment)*

Introduction

"In this workshop we shall do things but seem to do nothing.
We shall perform *symmetry operations*, geometrical transformations which leave an object exactly as it was before.

"The operations are *rotations* and *reflections* and we shall study these in 2-D and in 3-D, because we want to deal with the cube.

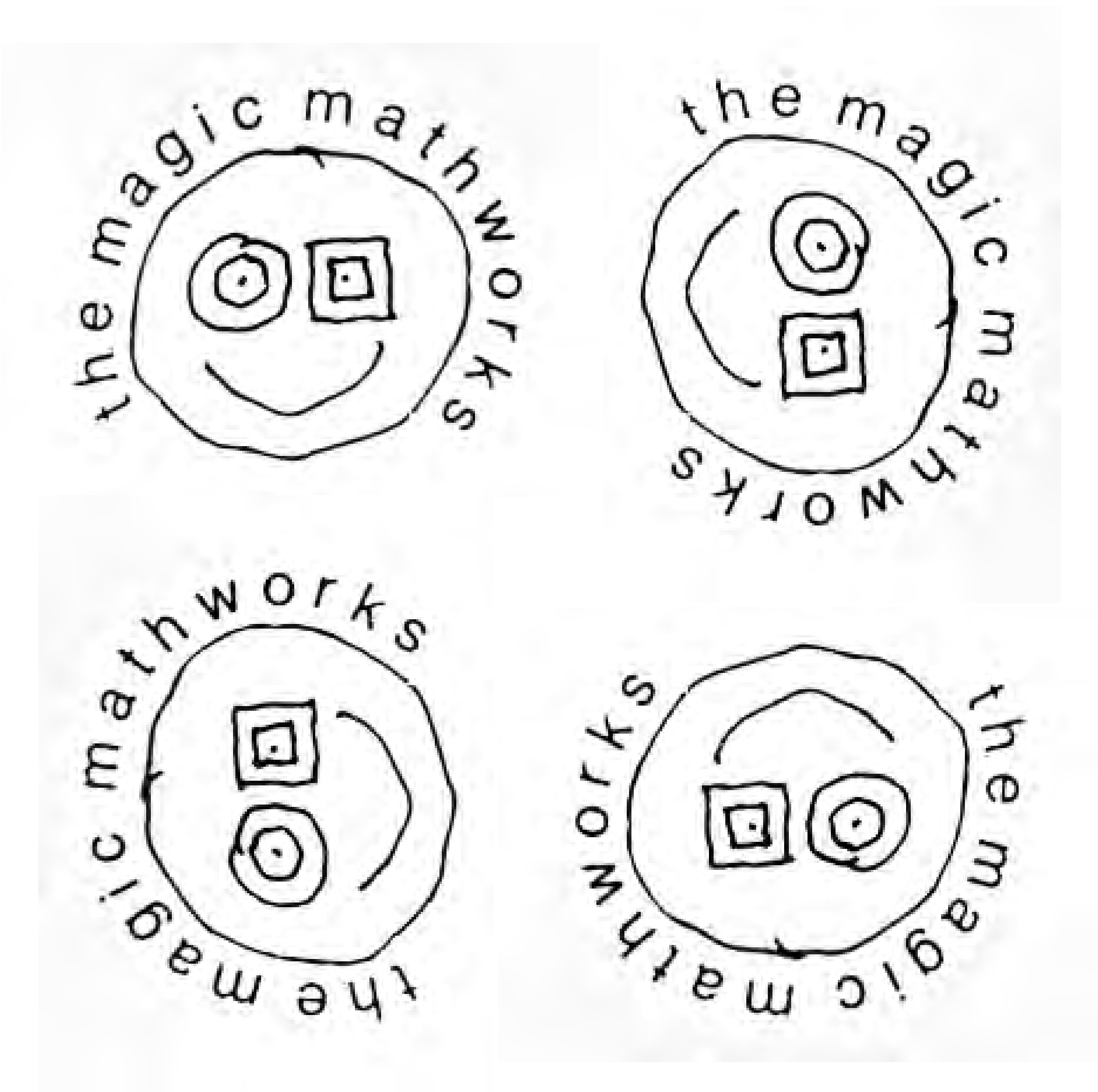
2-D symmetry: fitting an outline

Acetates A1-6 **E1** *Teacher demonstration*

Taking as motif the Magic Mathworks smiley face, invite the children to compare and contrast the symmetries displayed by designs with: rotation centres orders 2, 3, 4; 1, 2, 3 lines of symmetry.

the magic mathworks

the magic mathworks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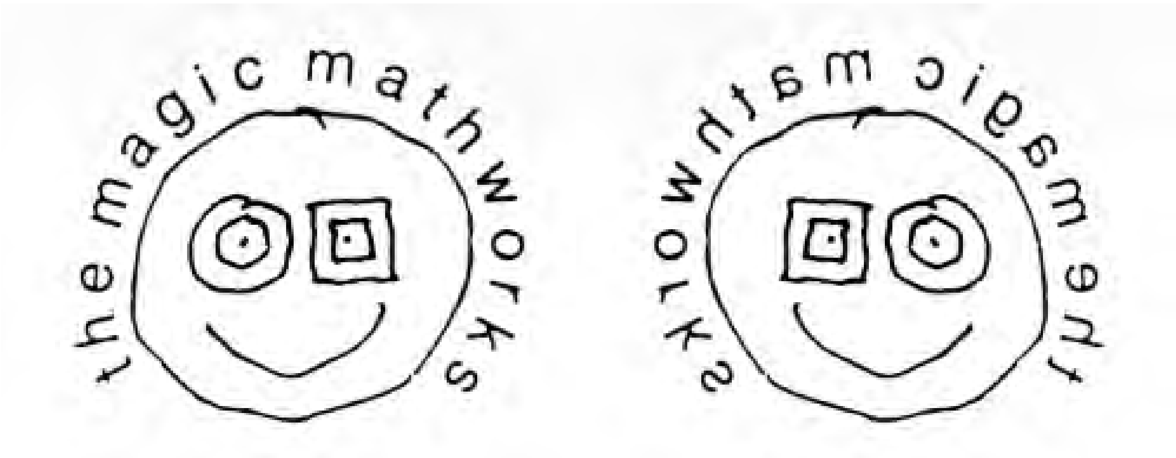
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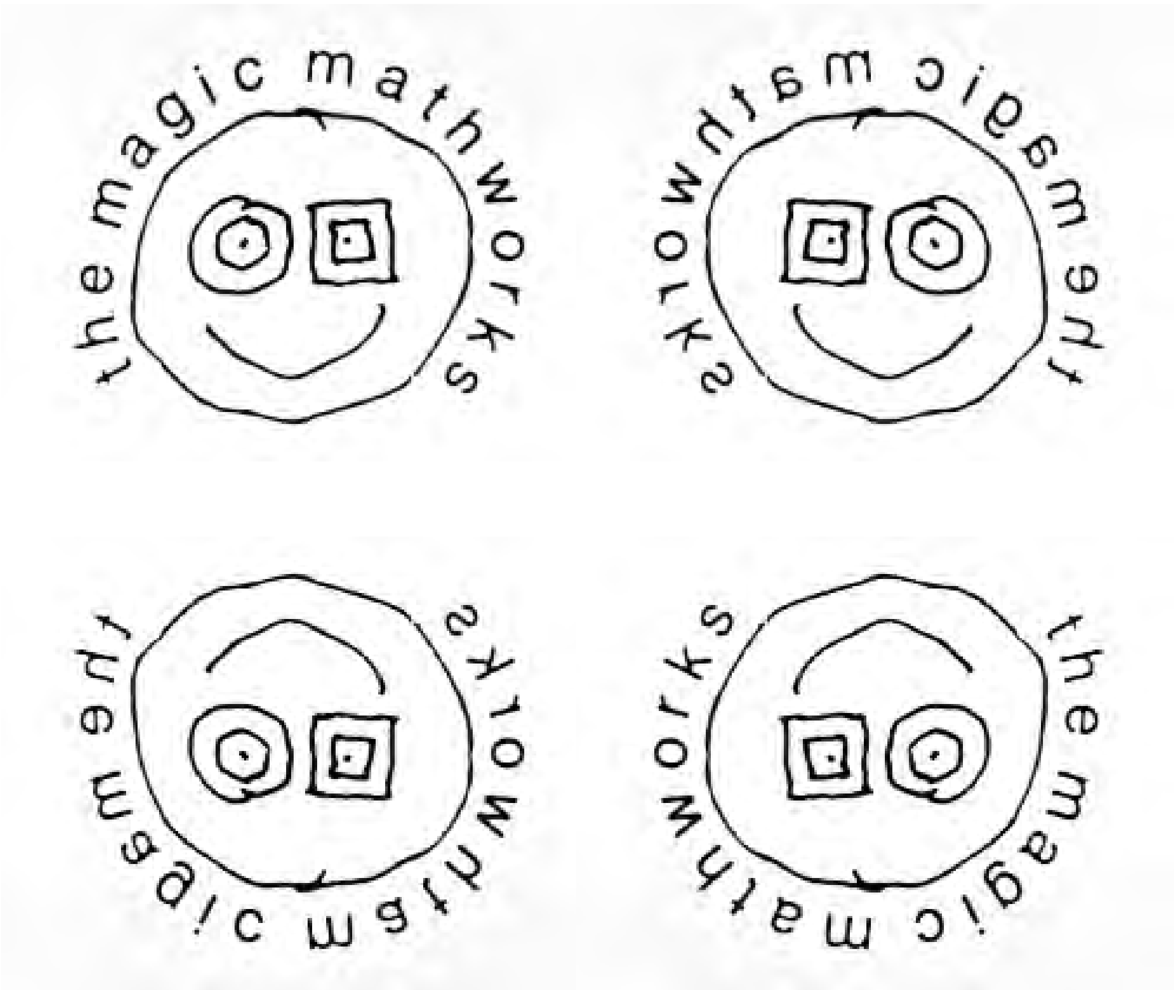
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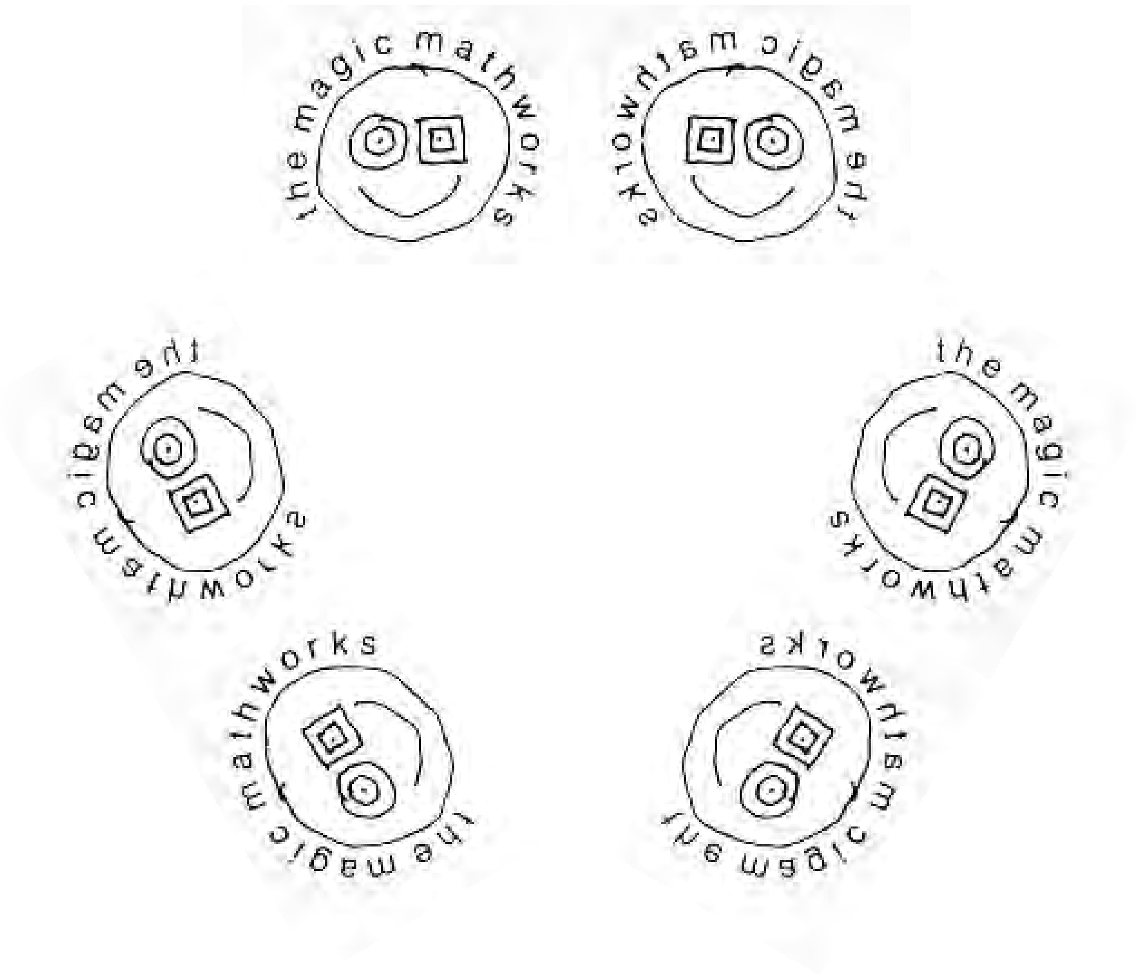
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[For a rotation you move the acetate in its own plane.
For a reflection you can't because you're trying to swap the front and back of the picture with respect to the mirror line. You can however give the picture a half turn about the mirror line in the 3rd dimension.]

Challenge the children to consider what this is going to mean when you come to 3-D objects.

Challenge the children to perform a half-turn rotation in the plane of the projector but using reflections. [Two perpendicular reflections are needed.] Point out the significance for future work that this angle is half the angle of rotation.

Draw the children's attention to a special property of designs with rotation symmetry order 2, half-turn symmetry: every 'image' point corresponds to an 'object' point in the direction of, and the same distance the other side of, a particular point, the *centre*. We can say it has been reflected in that point. We say the design has *central* symmetry.

As above, challenge the children to consider what this is going to mean when you come to 3-D objects.

Which of these symmetry *elements* are compatible, i.e. which symmetries can be found in the same design?

Acetate A7

Take suggestions and complete A7 by ticking possible cells from the blanks. [The correct entries are shown with a 'P' for 'possible'.]

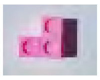
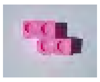
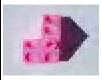
Number of symmetry lines / Order of rotation symmetry	0	1	2	3	4	5	6
1	**P**	**P**					
2	**P**		**P**				

3	P			P			
4	P				P		
5	P					P	
6	P						P

Order of rotation symmetry \ Number of mirror lines	0	1	2	3	4	5	6
1							
2							
3							
4							
5							
6							

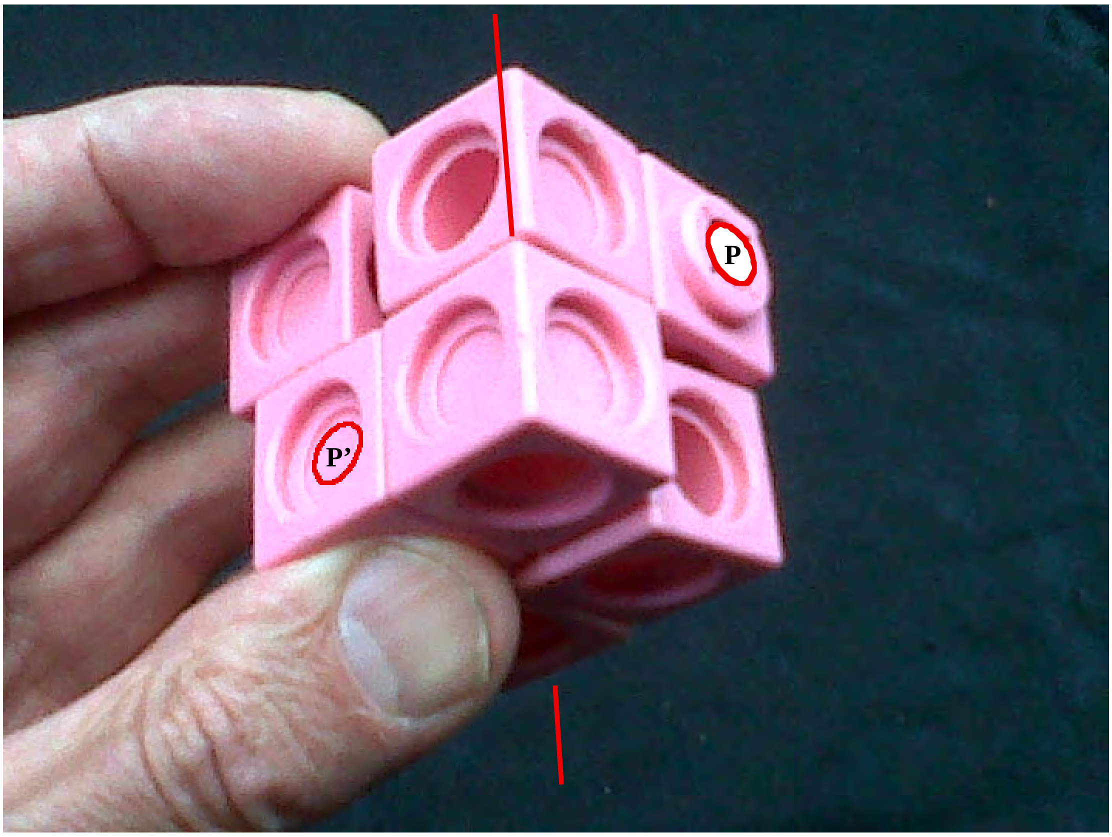
3-D Symmetry: fitting a hole

27 interlocking cubes, x 15	**E2** *Pupil experiment*
Demonstration Soma cube, 60 cm edge, x 1	The children copy the Soma pieces exhibited, using their interlocking cubes. Show how, in 3 dimensions, the analogue of a mirror line is a mirror plane; of a rotation centre, a rotation axis.
Acetate A8	Invite the class to count mirror planes and identify symmetry axes on each piece. Fill in the first two lines of A8, debating conflicting suggestions and challenging false allocations.

Soma piece:							
Mirror planes?							
Rotation axes?							
Centre?							

The following little experiment addresses the most controversial case: the 2-axis belonging to the 5th and 6th pieces.

The children make 2 copies of, say, the 5th piece and find they can fit them together to make a cube as shown. The lower piece provides a hole in which the upper can be located. A half-turn about the red axis takes the peg P to position P'.

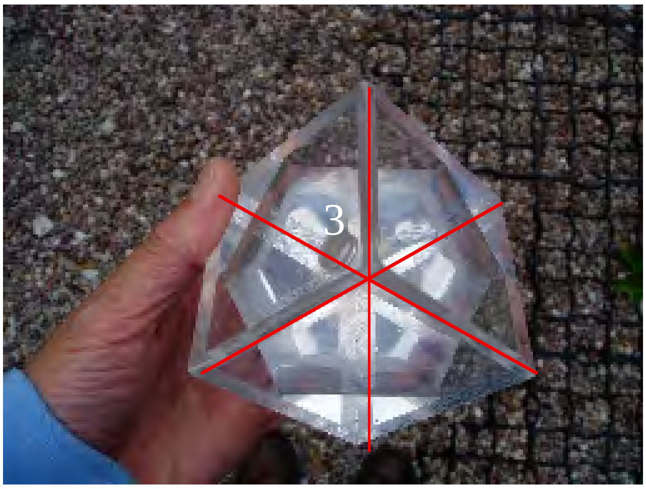


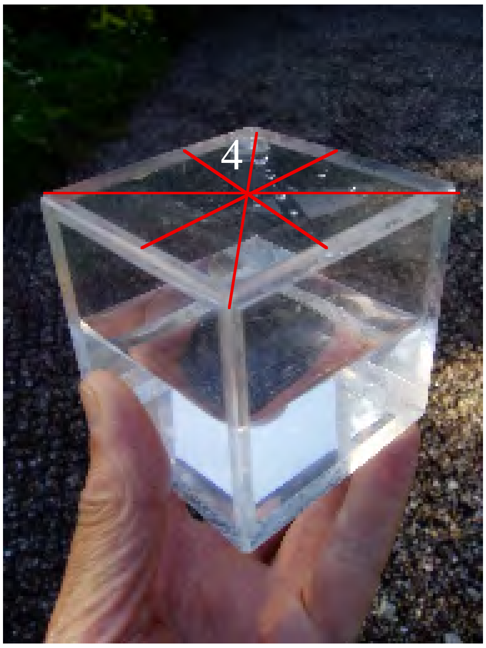
The cube's planes and axes

60 mm hollow perspex cube with pouring hole, x 15	**E3** *Pupil experiment*
	Ask the children to half fill their cubes. Invite them to orientate the cubes so as to produce different cross-sections as displayed by the water surface.
Measuring cylinder	Invite them to achieve:

with 108 ml line marked, x 1	a square, the largest possible rectangle, a regular hexagon;
Jug as water supply, x 1	and in each case identify the vertical symmetry planes and rotation axes.

The red lines indicate the vertical symmetry planes, the white numbers the orders of rotation symmetry of the vertical axes.





Destroying the cube's symmetry

8 blue interlocking cubes, x 15, stock of pink and green cubes to be raided as required

E4 *Pupil experiment*

Issue this challenge:

1. Use your blue cubes to make a cube 2 x 2 x 2.
2. *Either* by plucking individual cubes out of your model *or* by swapping red – and also, if necessary, green – cubes for blue cubes, try to remove all the cube's symmetries. The challenge is to do so by removing or swapping as few cubes as possible.

[It is not possible to destroy the symmetry of a cube 2 x 2 x 2 by removing cubes – not so for one 3 x 3 x 3 incidentally. Two additional colours are needed: 2 of one colour must be swapped along an edge, 1 of a second colour at a corner such that all 3 share one face. The point of the challenge is to show how easy it is to miss symmetry elements.]

The order of rotation symmetry of the cube

E5 *Pupil experiment*

Ask this question:

How many ways can we turn a cube so that it fits the hole in which it sits (the *order* of rotation symmetry of the cube)?

The children are to lay a hand on the table and make a right angle between thumb and forefinger. This will be the hole in which they are going to fit their cube.

40 mm beechwood cube, x 15

1. Ask the children to choose one of the cube's faces to lie on a matching face in the corner cube. Show that the cube can be turned 4 times to fit.
2. Ask them to choose one of the cube's vertices to lie in a matching vertex of the cubic hole (the junction of thumb and forefinger). Show that the cube can be turned 3 times to fit.
3. Ask them to choose one of the cube's edges to lie in a matching edge of the corner cube (the thumb). Show that the cube can be turned 2 times to fit.

Acetate A9

Complete A9 together.

We can stand the cube any of		ways round on any of its		faces.	The order of symmetry is therefore	x =	
				vertices.		x =	
				edges.		x =	

We can stand the cube any of	**4**	ways round on any of its	**6**	faces.	The order of symmetry is therefore	**4** x **6** =	**24**
	3		**8**	vertices.		**3** x **8** =	
	2		**12**	edges.		**2** x **12** =	

2 x 2 x 2 cube with diagonals (viz. pairs of opposite vertices) colour-coded, x 15

E6 *Pupil experiment*

Tell the children that they're now going to make the same count in a different way.

Remind the children that a space diagonal is a line joining opposite vertices. Tell them that their 4-colour cube has been made by giving each space diagonal a different colour.

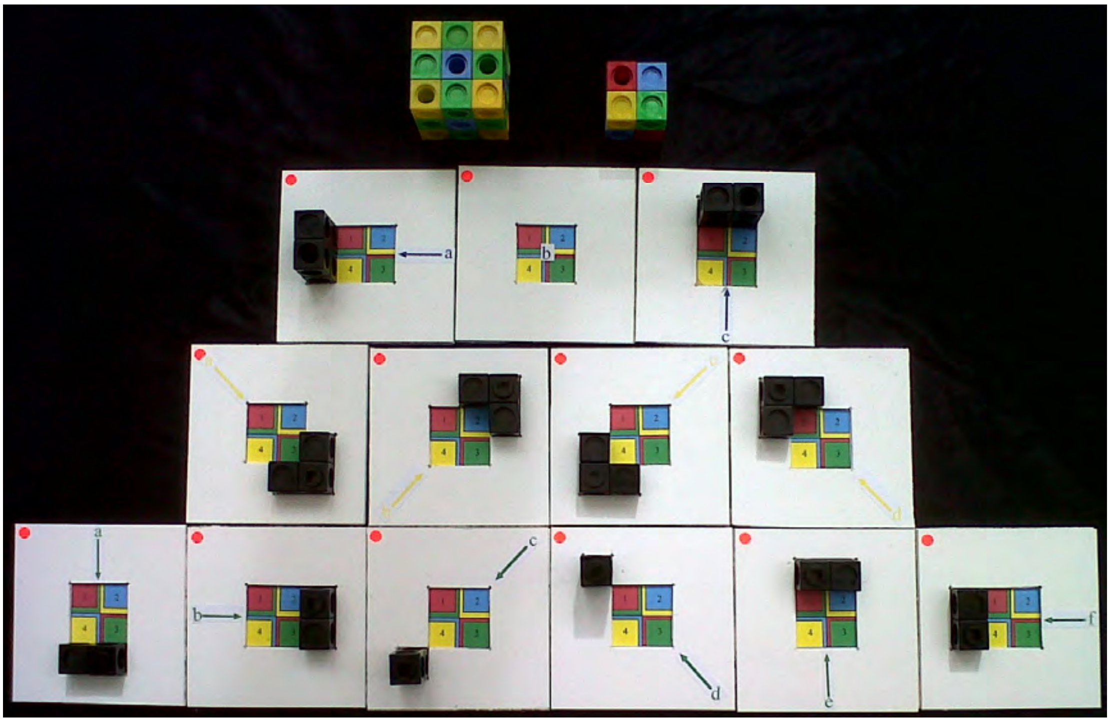
3 x 3 x 3 cube with rotation axes colour-coded, x 1

Indicate on the demonstration cube the axes which they have been using in **E5**.

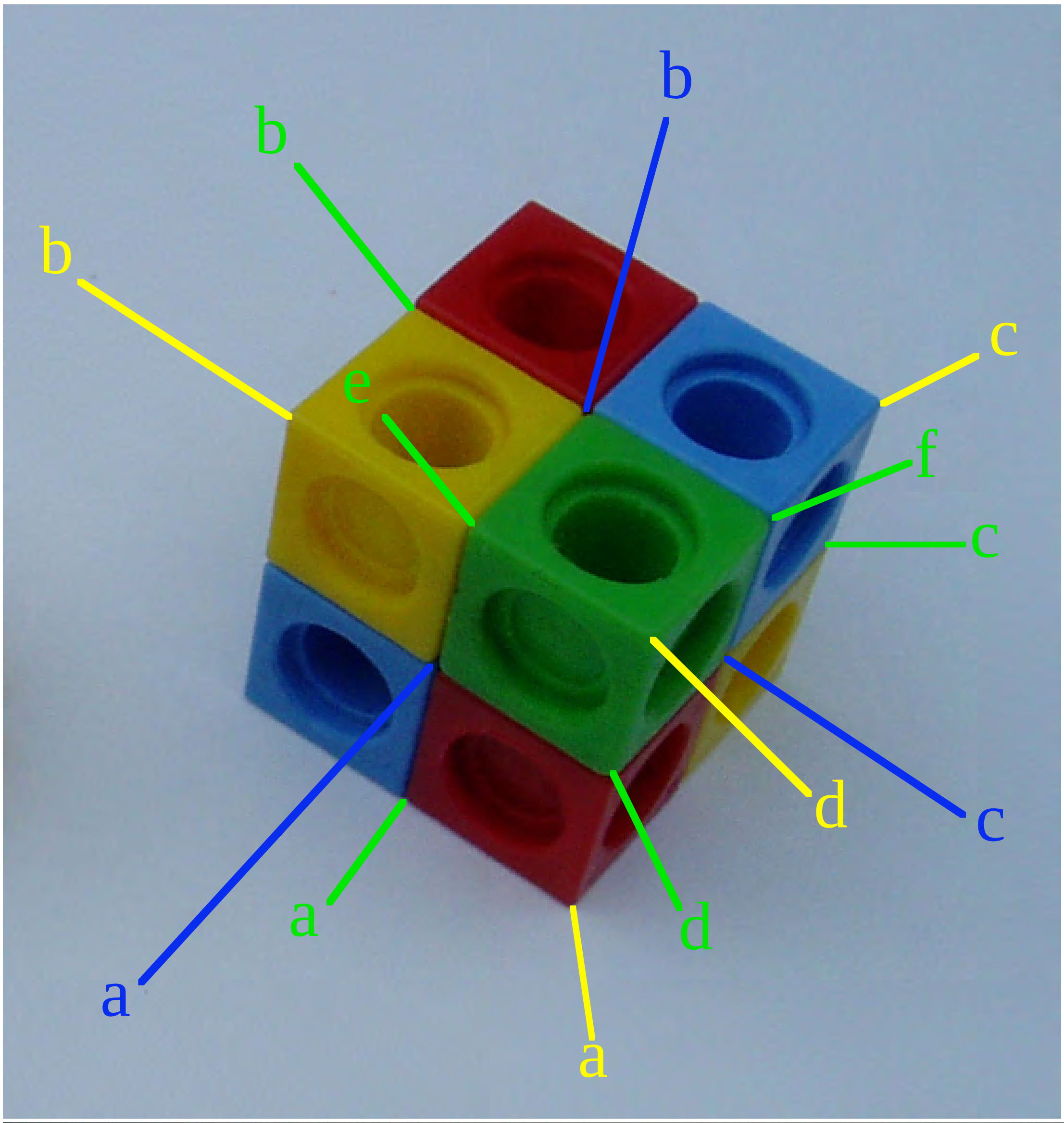
Locator for each of the 13 axes, x 1

Give each pair their individual locator.

The picture below shows the demonstration cube and, alongside it, one of the 13 4-colour cubes. In the 3 rows below are respectively the locators for the 3 4-axes, ('blue' a,b,c), the 4 3-axes ('yellow' a,b,c,d), the 6 2-axes ('green' a,b). The black cubes serve the function of the hand in **E5** but this time identify each separate axis. Towers 2 cubes high indicate that the axis is horizotal; towers 1 cube high suggest that it is inclined down towards them. The coloured squares in the cutout, numbered 1 to 4 clockwise from red, represent the top face of the 4-colour cube.



Interlocking cubes in red, blue, green, yellow taken by the children as needed	Instruct the children as follows (or in words to the same effect) "From the starting position you must turn your cube about the axis you have been given. Turn it clockwise looking into the cube, i.e. along the arrow. Each time it arrives in a position in which it fits the hole, note the cubes which fall in positions 1 to 4 and build a stick 4 cubes long using the same colours. Build your stick from left to right. The peg which sticks out of each cube must point right."
Receiving frame for cube sticks, x 1	Ask the children how many sticks they should end up making, given the type of axis they have been allocated. Stress that they must lay out their sticks in the order in which they are made.
Acetate A10	Show A10 to explain the notation with which each entry is labelled in the receiving frame.



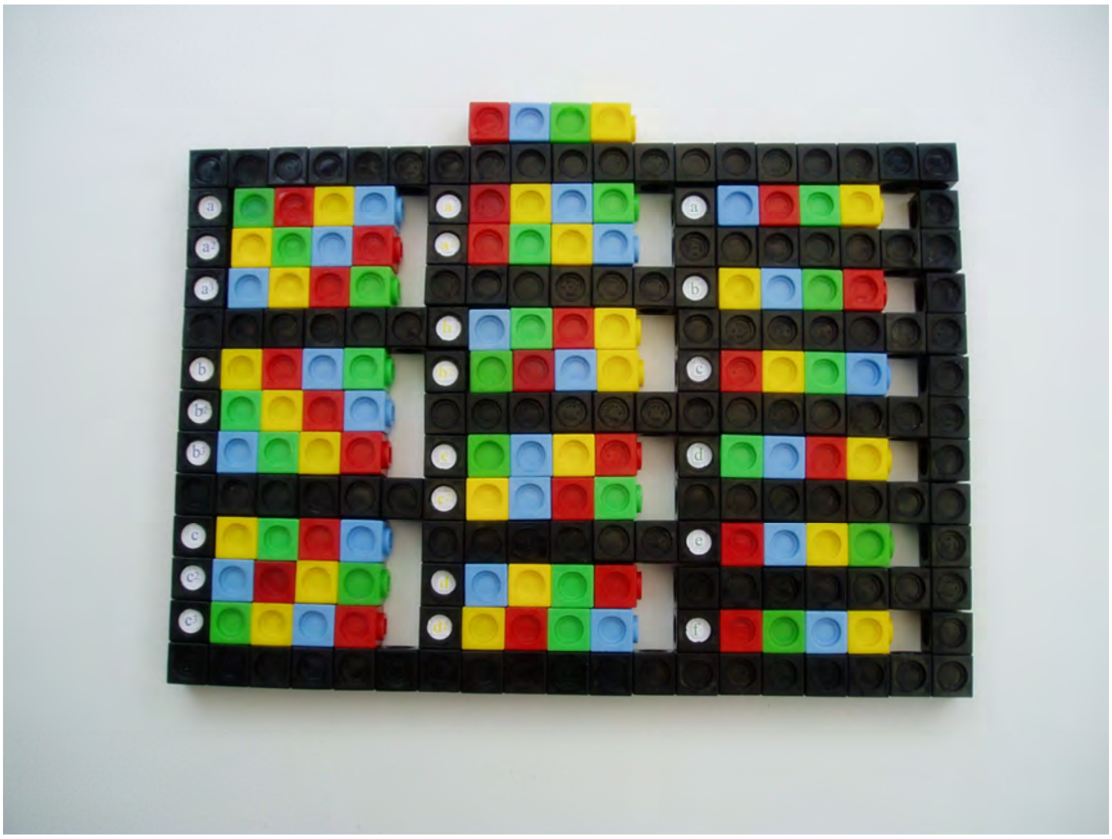
Make all turns clockwise looking into the cube.

a, a^2, a^3, a^4 = 1/4, 2/4, 3/4, 4/4 of a turn about that axis, so a^4 brings you back to the start and is therefore just the identity operation, e.

a, a^2, a^3 = 1/3, 2/3, 3/3 of a turn, so $a^3 = e$.

a, a^2 = 1/2, 2/2 of a turn, so $a^2 = e$.

As the sticks are completed, collect them in and set each in its appointed place in the frame, displayed vertically at the front of the room.



Acetate A11 Have A11 on the projector throughout to display the slot labels.

a	
a^2	
a^3	

b	
b^2	
b^3	

c	
c^2	
c^3	

a	
a^2	

b	
b^2	

c	
c^2	

d	
d^2	

a	

b	

c	

d	

e	

f	

Ask the children to comment.

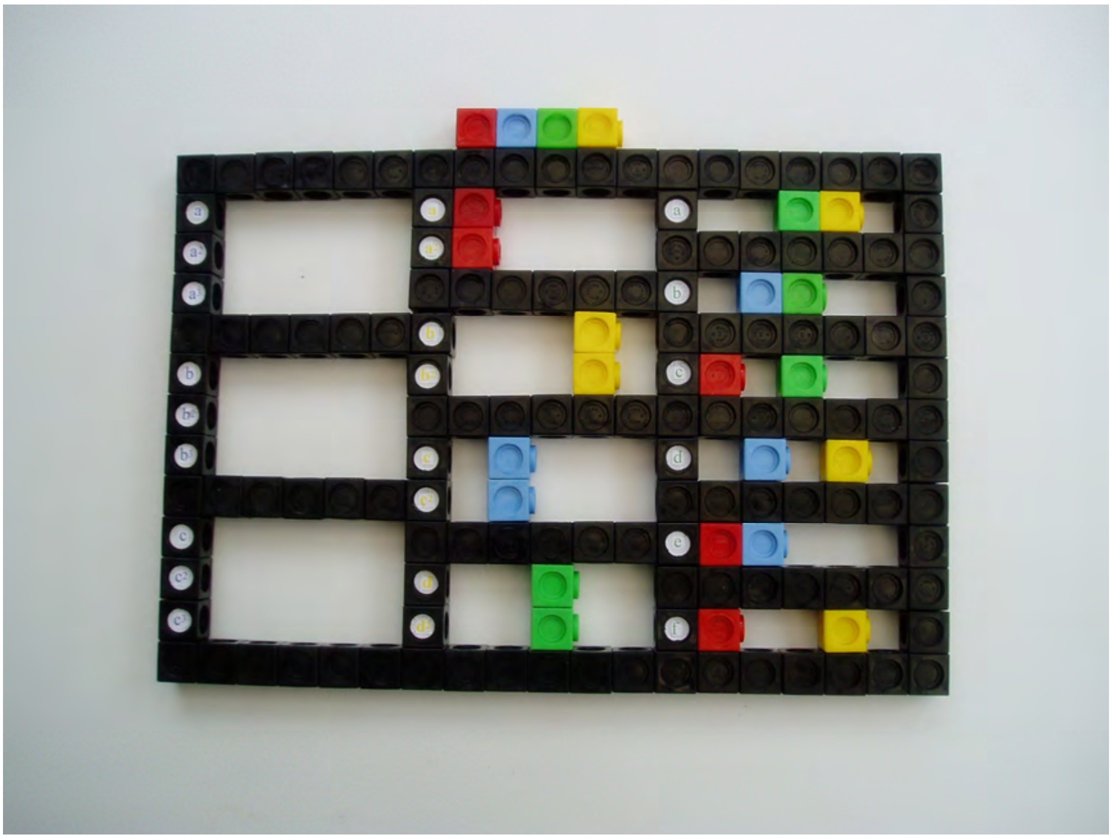
[All the sticks are different.

The number of sticks is 1 + 3(3) + 4(2) + 6(1) = 24.

In a single box in the first column, no colour stays in the same position – because rotation about a 4-axis moves all the diagonals;

in a single box in the second column, one colour stays in the same position – because the diagonal aligned with the symmetry axis stays put;
in a single box in the third column, two colours stay in the same position – representing the diagonals which do not share an edge with the 2-axis concerned.]

Acetate A12 Show A12.



Point out to the children that there are 4 x 3 x 2 x 1 or 4! = 24 ways of arranging the 4 colours. Investigate why with the help of the children. Tell them that an arrangement is called a *permutation*, and to arrange things in a different way is to *permute* them. Point out that what the children have succeeded in doing is permuting the 4 space diagonals of the cube in all possible ways.

Combining rotations

As **E6** — **E7** *Pupil experiment*

Having looked at rotations about different axes separately, announce that the time has come to combine them in every possible way.

Ask a pair to repeat one of their rotations.
Take their card with the cube as it ends up and transfer the cube in the same orientation to the card of another pair.
Ask *them* to repeat one of *their* rotations on the cube they have just received.
Ask them to make a stick to record the result.
Ask them to match it with one already in the receiving frame.
Point out that they have performed two rotations in succession and that this has had the effect of another rotation.

Now repeat that sequence but *in reverse*.
Ask the children whether they expect the same result. [The same result only occurs if the same rotation axis is used twice.]

Ask the children to imagine you going round the tables, each time collecting the result of the first rotation, but always passing the cube to the same second pair, till you had repeated the experiment all 24 times.

Acetate A13 — Show A13 and explain that you could record the result of each of the 24 experiments by filling in a box in A12 till you had completed a row.

Combining rotations of the cube

An entry in the table shows the effect of performing the operation shown along the top followed by the operation shown down the side.

	e	a	a^2	a^3	b	b^2	b^3	c	c^2	c^3	a	a^2	b	b^2	c	c^2	d	d^2	a	b	c	d	e	f
e																								
a																								
a^2																								
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Tell them that each result would be different so that you would end up with 24 symbols, each occurring once.

Tell them that the same would apply to every row and every column. Ask the children where they have seen a table like that before. [In a Sudoku puzzle]. Tell them that such a table is called a *Latin* square.

Tell them that mathematicians call a collection of objects a *set*, of which each object is an *element*, that the 24 rotations are the 24 elements of a special type of set called a *group*, and that the Latin square is the *operation table* for the group, in this case the *rotation symmetry group of the cube*, which therefore has *order* 24.

The cube's reflective symmetry

Tell the children that the time has come to think about what reflection means in 3 dimensions.

8-colour cube, viz. 4-colour cube with opposite vertices distinguished, x 15

The same 8 cubes loose, x 15

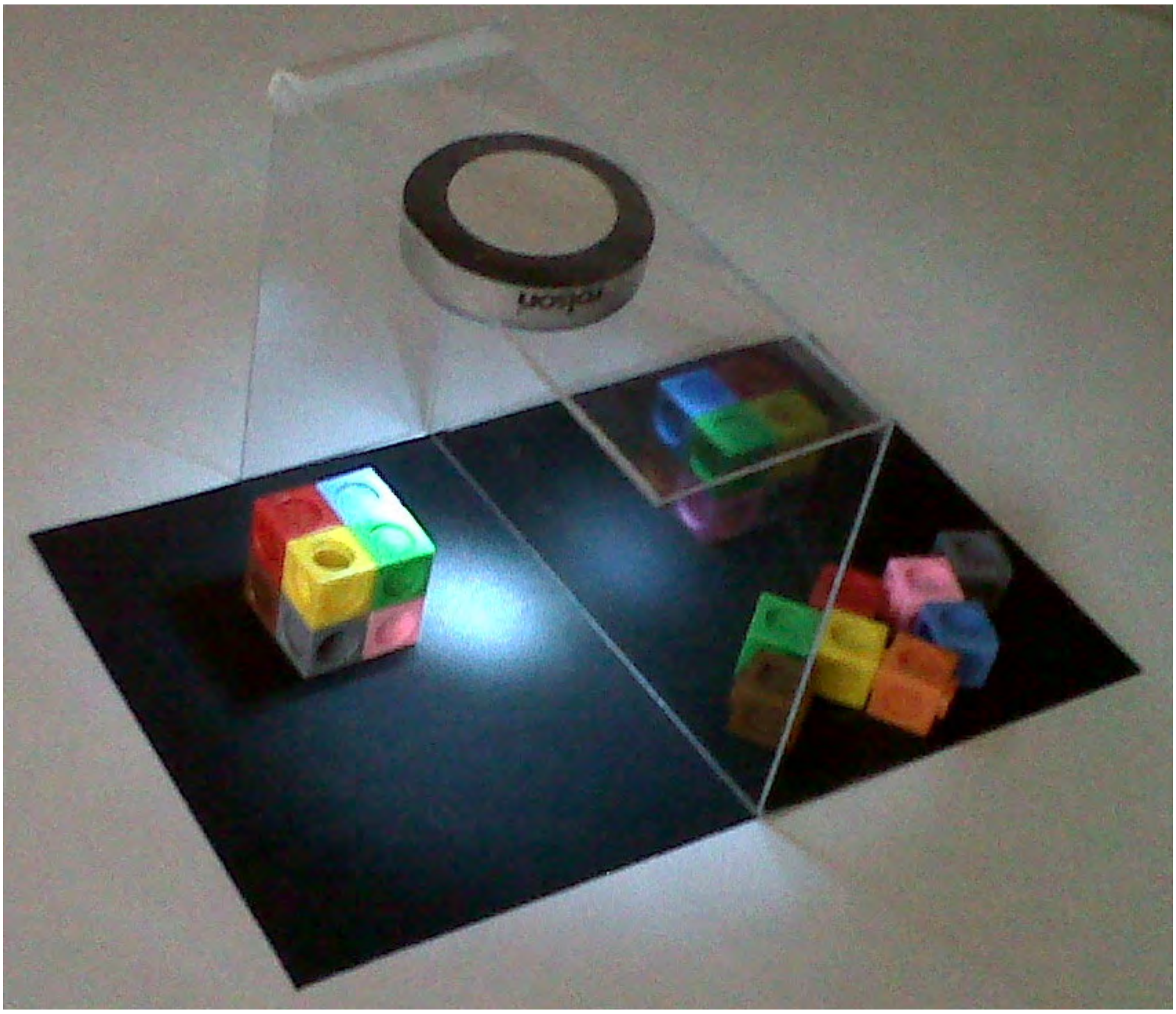
Semi-silvered mirror unit, x 15

E8 *Pupil experiment*

The picture below shows the 8-colour cube to the left of the semi-silvered mirror, in which the reflection can be seen. The children must use the loose cubes to build the mirror form and bring it into coincidence with its reflection.

Tell them that the mirror form is called an *enantiomorph*.

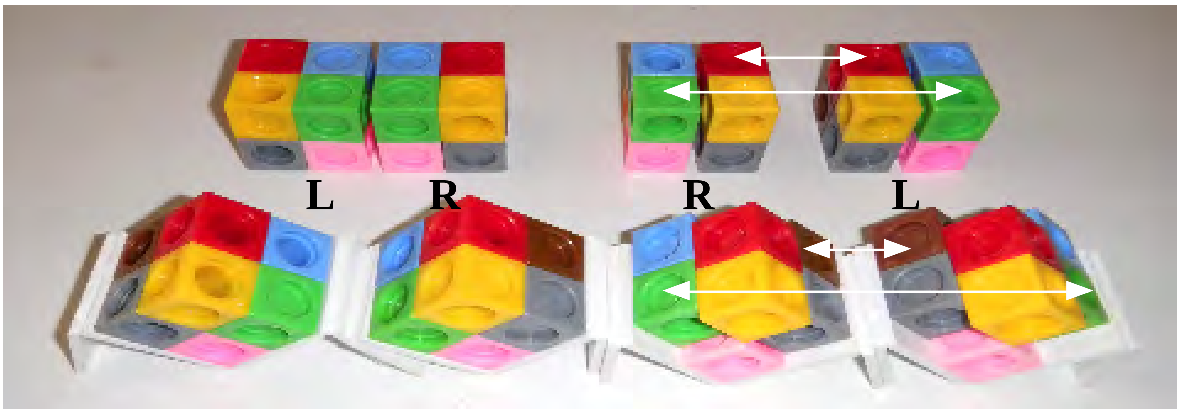
Ask them if they can turn the two cubes so that they look the same. [Answer: No: they are not *superimposable*.]



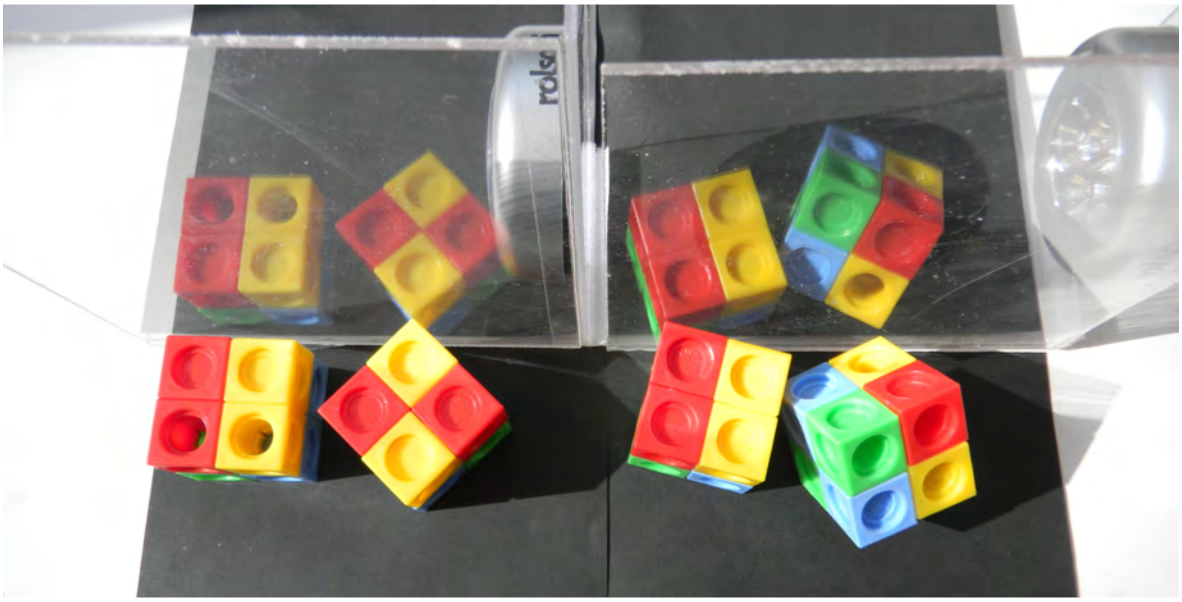
What happens when you reflect something in a mirror?

Ask them to recall what happens in 2-D: you reverse front and back to get the reflected form by turning the picture over. This involves a rotation in the *3^{rd}* dimension about the mirror *line*. How can you perform a rotation in the *4^{th}* dimension about the mirror *plane*?
[Answer: by turning the cube *inside out*.]
Promise the children that they will shortly do that by building a hollow cube but that they could immediately swap front and back of their 8-colour cubes. Remind them that the cube itself has two kinds of symmetry plane: one parallel to a pair of faces, the other running through opposite edges.

Acetate A14 — Project A14 to show the corresponding ways of doing it. The pictures show the cubes which must be swapped to get back from the right-handed form (**R**) to the left-handed form (**L**).


L
R
R
L

Acetate A15   Show A15. Point out that these two cubes use 4 colours. Ask the children how they differ from the 4-colour cubes they have been using. [These have planes of symmetry.] Ask them how they have been arranged on the left and on the right. [On the left their symmetry planes are parallel to the mirror; on the right, not.]



Ask the children to say whether these two cubes are superimposable or not. [They are.] Ask them whether it matters how they are arranged in front of a mirror. [It doesn't.] Ask them why not. [You can always turn the reflected cubes to be the same way round as the originals.]

A question to leave hanging: "Does an object have to have a mirror plane to be superimposable?"

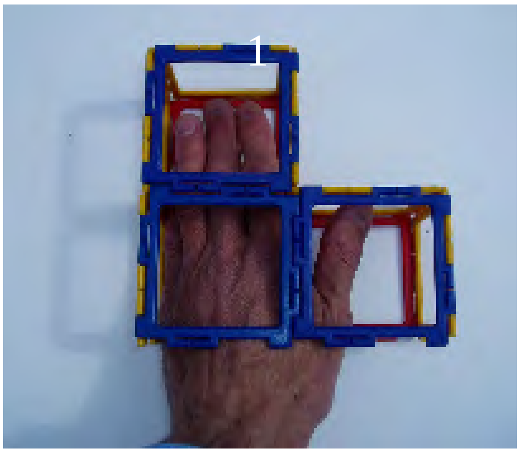
Polydron Framework interlocking squares, as needed

Acetate A16

E9 *Pupil experiment*

Tell the children the time has come to turn things inside out.

They are to check whether or not a left-hand glove is superimposable. Show A16 and ask the children to perform the experiment.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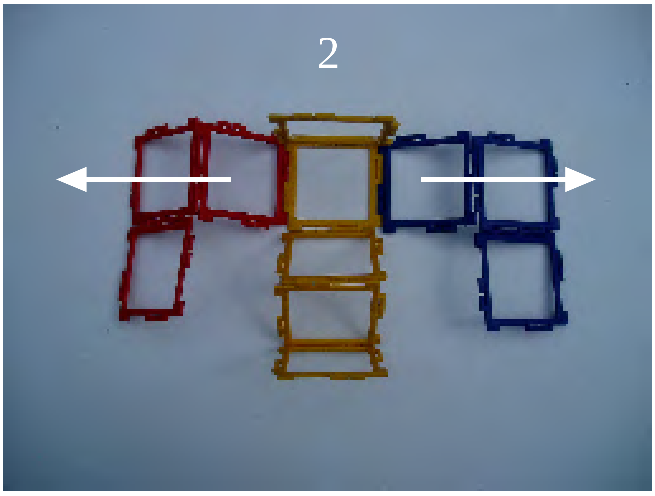
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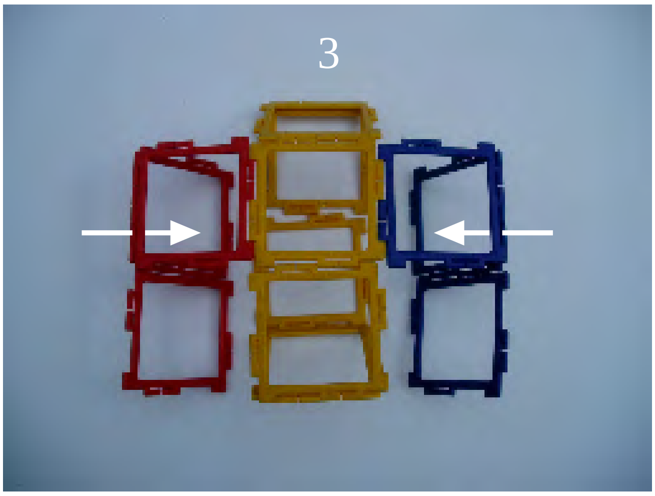
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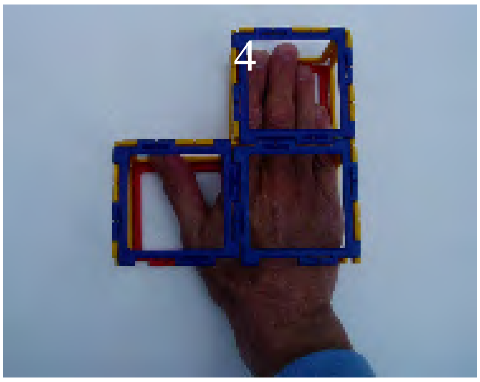
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3-D objects with central symmetry

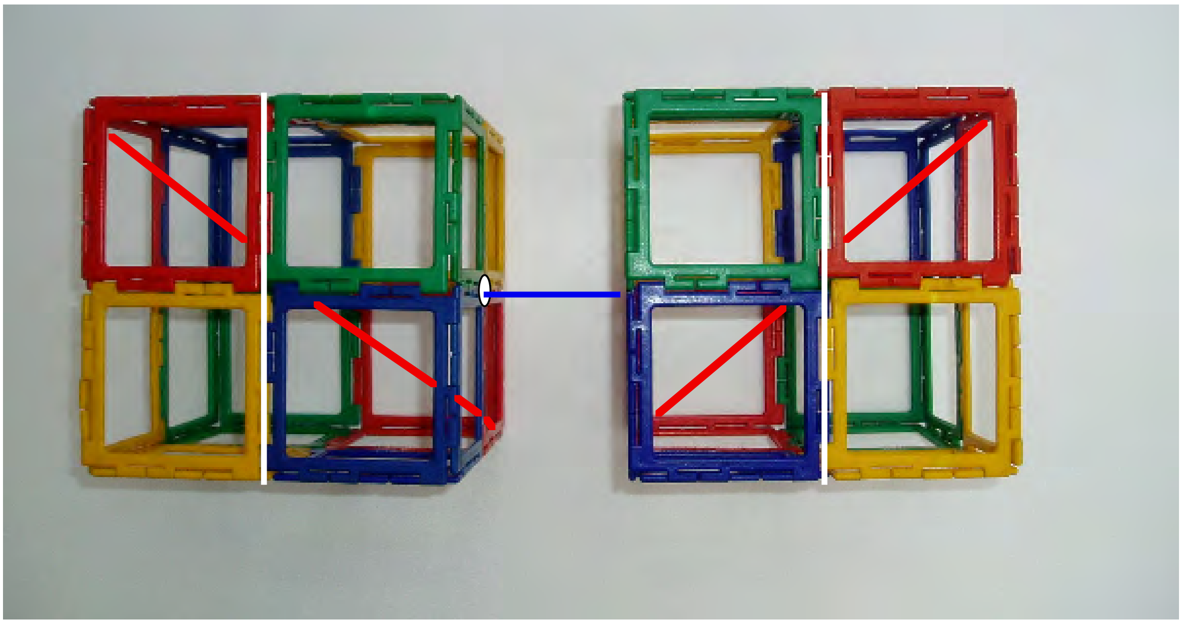
Acetate A17 **E10** *Pupil experiment*

Now the children are to build their original 4-colour cube, but in a hollow version, and turn *that* inside out.

Ask them to start with the trade mark ‘Polydron’ on the inside. They’ll know if they’ve succeeded in turning their model inside out if it ends up on the outside.

Show A17.
Ask them how to get from frame 4 to frame 5.

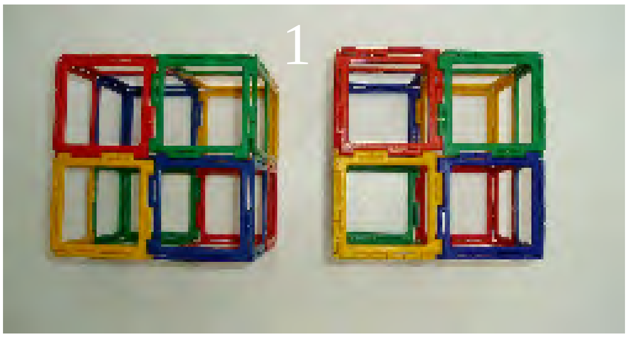
The answer is to give the mirror form a half-turn about an axis perpendicular to the mirror, shown by the blue line.



Point out to the children how, for example, the red diagonal ends up parallel to the original direction.

Ask the children whether or not the 4-colour cube is superimposable.
[They have just demonstrated that it is.]
Ask them if the 4-colour cube has a plane of symmetry.
[It does not.]
Ask them what they conclude.
[The answer to the question left hanging at the end of **E8** is ‘No’.]

Ask them if, nonetheless, there’s something special about the 4-colour cube for it to behave in this way.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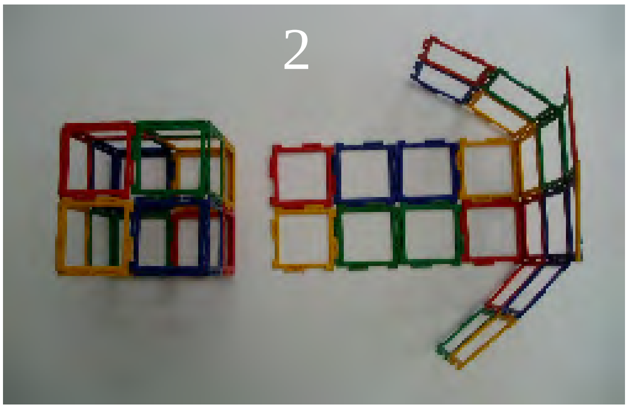
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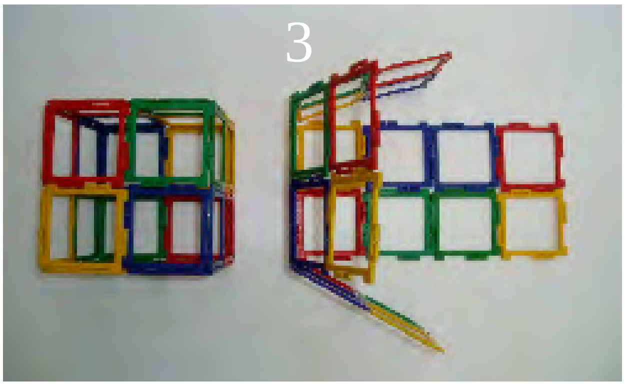
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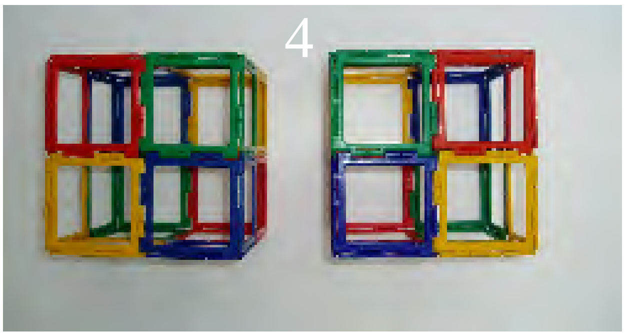
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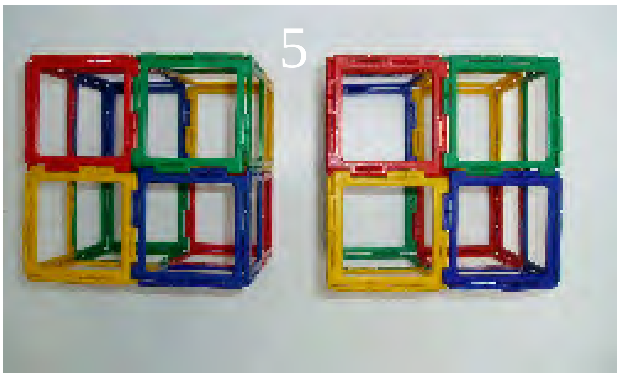
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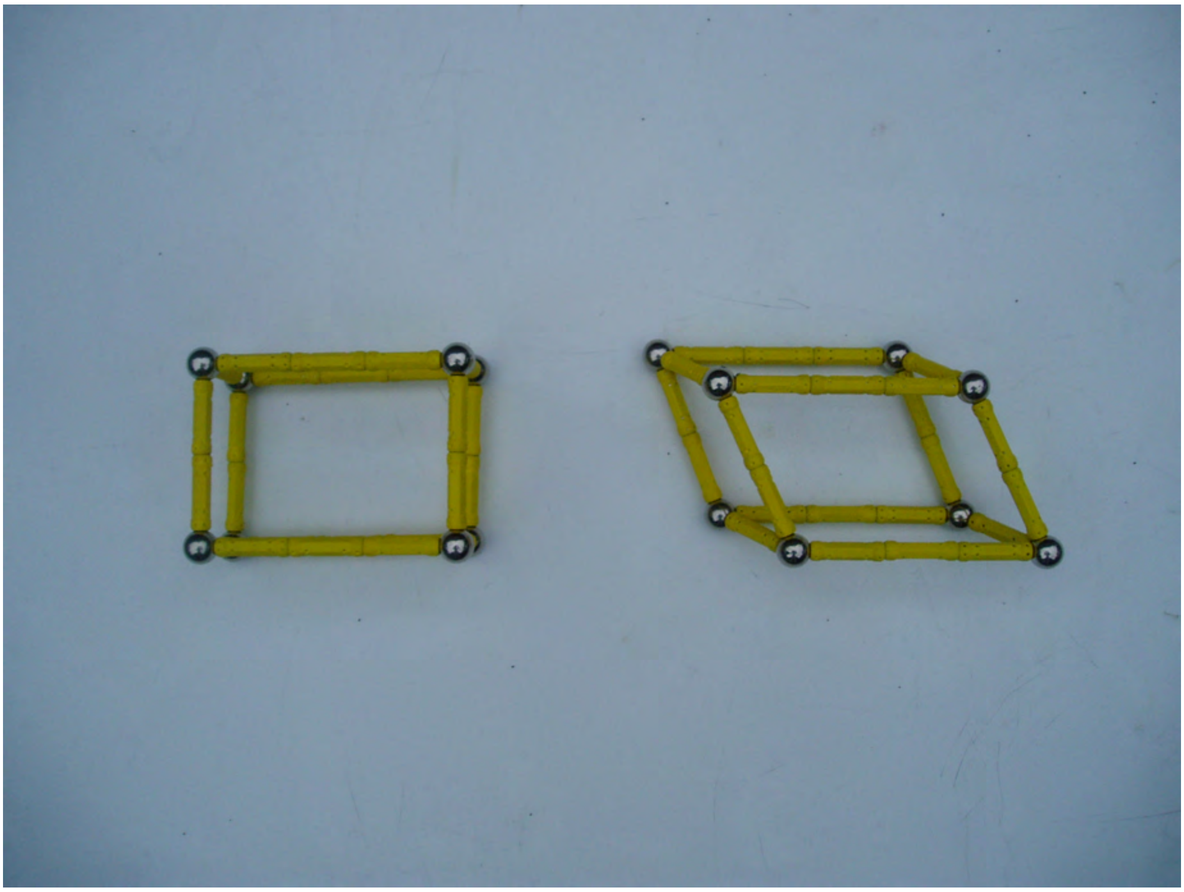
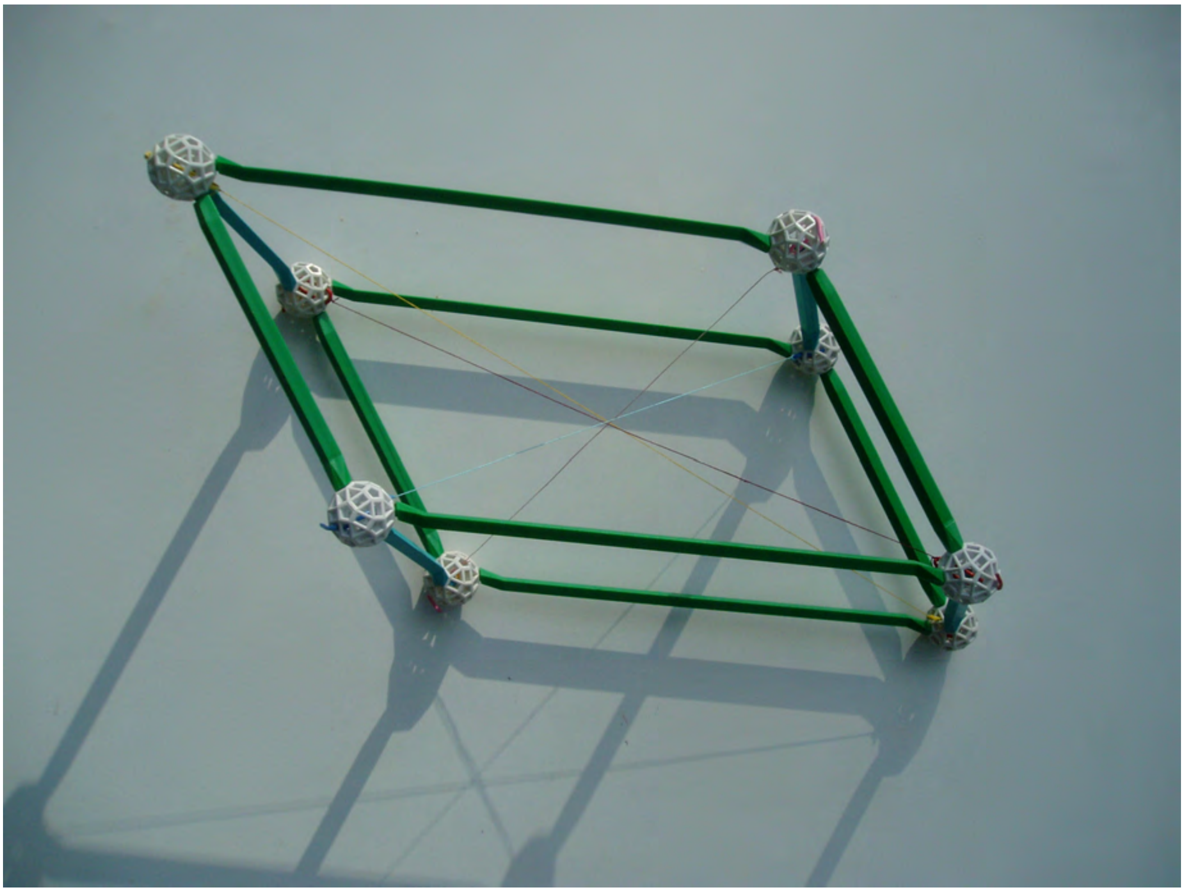
Deformable cuboid in Geomag, x 1 General parallelepiped model in Zome, space diagonals picked out in cotton, x 1	**E11** *Teacher demonstration* Use the Geomag model to show how, considered as a linkage, a cuboid can be deformed into a general parallelepiped. Ask the children to list the symmetries which are destroyed in the process. Ask them to describe the resulting shape. [The 6 faces which began as rectangles are now general parallelograms. But opposite faces are still congruent.] Exhibit the Zome model, which copies the deformed cuboid. Tell them it might represent for example a crystal of the mineral calcite.





Ask the children to look at the cotton threads, (whose colours match the coding of their 4-colour cubes), and cast their minds back first to their hollow 4-colour cubes (**E10**), then all the way back to their very first experiments with the smiley faces (**E1**). [In all these cases there is a special point, the *centre*, equidistant from ‘antipodal’ points on the object, i.e. they all have *central* symmetry.]

Recap the situation with their reflected 4-colour cubes: they are superimposable by virtue of their central symmetry. An object may be superimposable either because it has a plane of symmetry, or because it has central symmetry, or – in the case of the cube itself – both.

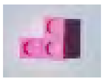
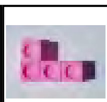
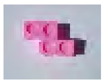
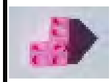
Acetate A8

E12 *Pupil experiment*

Multilink Soma models from **E2**

The time has come to complete the 3rd line of the table on A8, that asking about central symmetry.

The table should be completed as follows.

Soma piece:							
Mirror planes?	1	1	1	1			3
Rotation axes?	1 (order 2)		1 (order 2)	1 (order 2)	1 (order 2)	1 (order 2)	1 (order 3)
Centre?			Yes				

Semi-silvered mirror unit, x 15

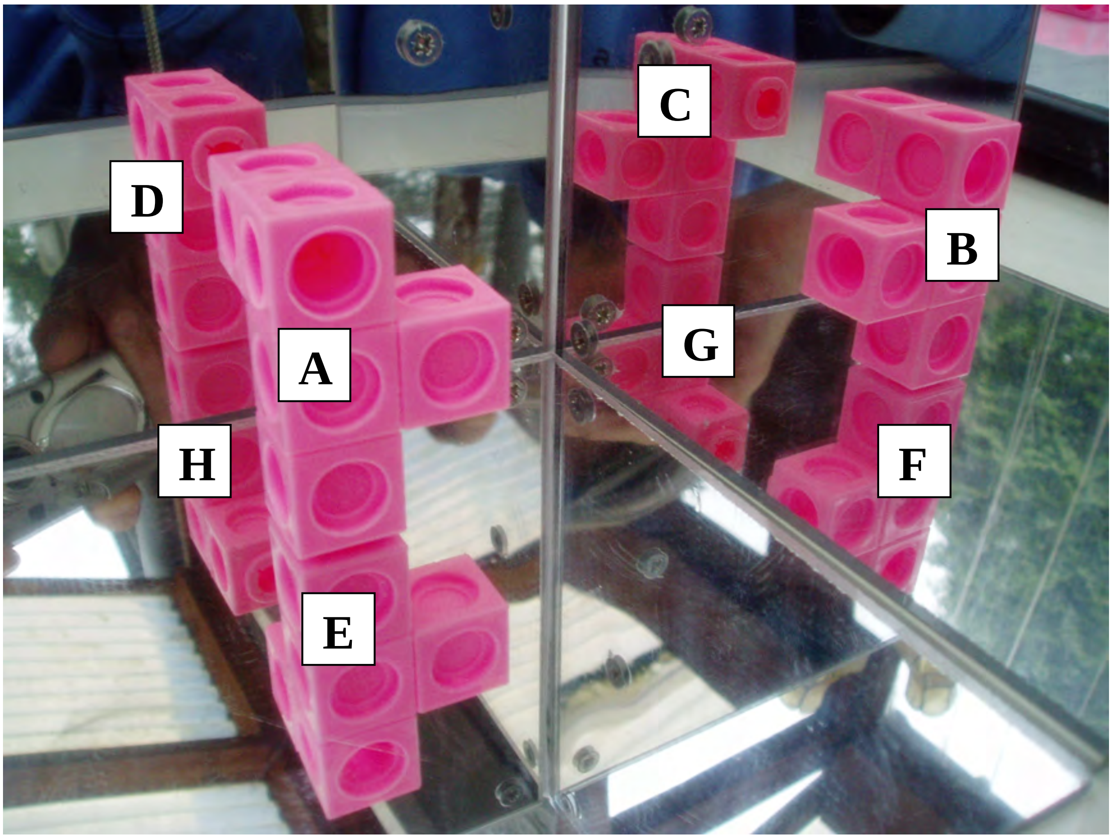
E13 *Pupil experiment*

Show A18. Ask the children what they think the picture shows. [A model made from 5 cubes, having no symmetry, reflected in a 'mirror cube' (a set of 3, mutually perpendicular mirrors).]

Loose cubes as needed

Ask them to consider models made from the 8 parts they see.

Acetate A18


C
D
B
A
G
H
F
E

Ask:

1. “What symmetries has the whole model ABCDEFGH?”
[All the symmetries of a cuboid.]

2. “What symmetries has the part ABCD?”
[2 mirror planes and a 2-fold rotation axis along their line of intersection.]

3. “What symmetries has the part AB?”
[1 mirror plane.]

4. “What symmetry has the part AG (or BH or CE or DF)?”
[Central symmetry alone.]

5. “Can you make up another imaginary model which has only central symmetry?”
[ACFH or BDEG.]

Challenge the children to build models having only central symmetry. They should build a pair of each. They may then use their semi-silvered mirror unit to arrange one as the reflected from of the other. They should then give one model a half-turn about an axis normal to the mirror. If they then remove the mirror, they should see that their model is superimposable (as in the transition from frame 4 to 5 on A17).

Use the following activity only if time allows.

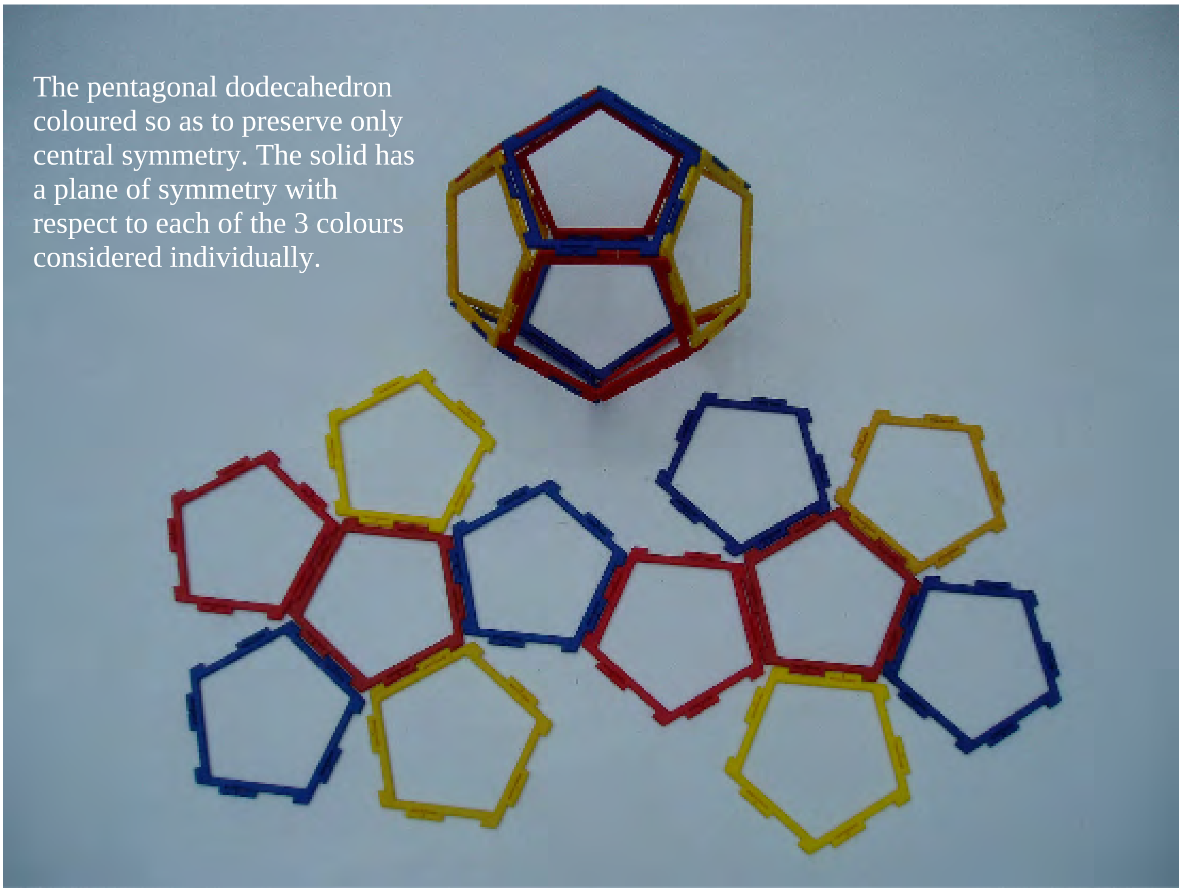
Polydron triangles, pentagons as needed

E14 *Pupil experiment*

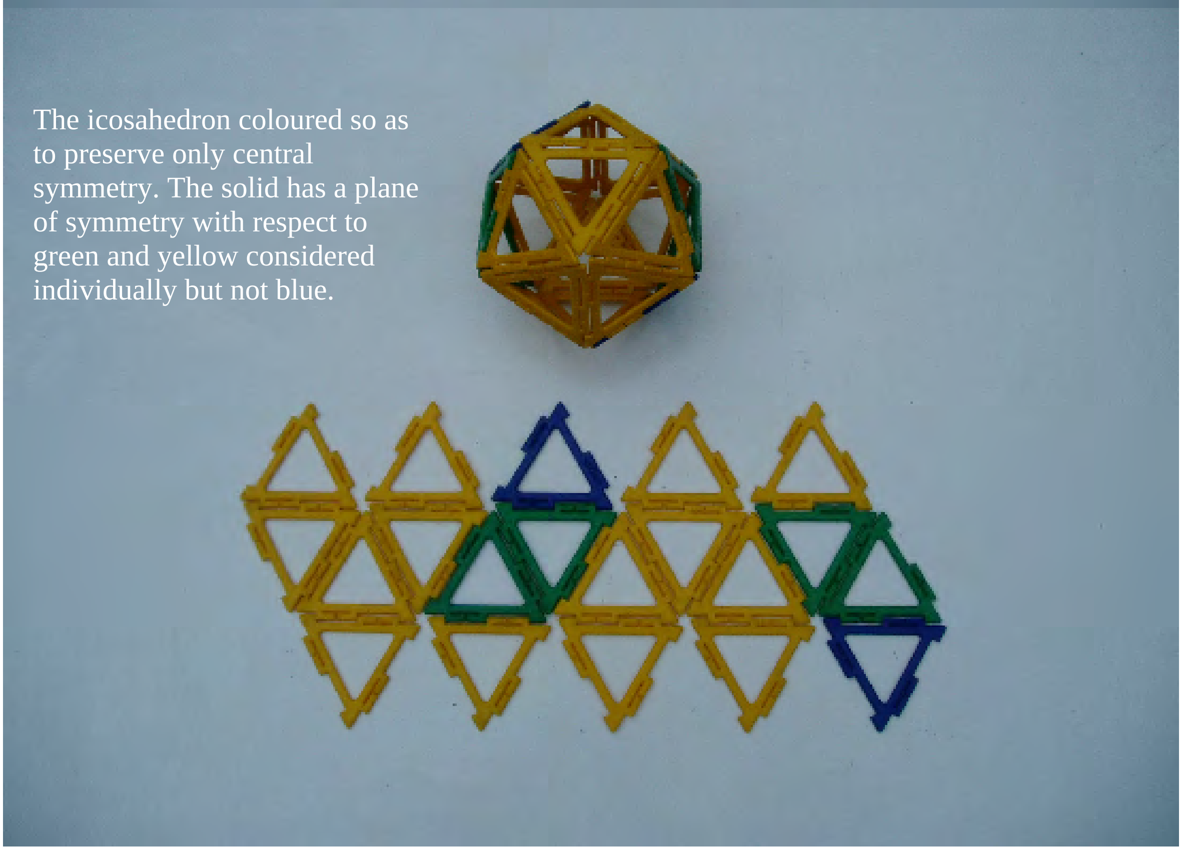
Exhibit a complete regular dodecahedron in yellow. Point out that it has 12 faces and that 3 meet in every vertex. The challenge is to build the model but swap yellow for other colours so that the children are left with a regular dodecahedron which has only central symmetry.

Exhibit a complete regular icosahedron in yellow. Point out that it has 20 faces and that 5 meet in every vertex. The challenge is to build the model but swap yellow for other colours so that the children are left with a regular icosahedron which has only central symmetry.

Here is the most economical way to achieve this in each case.



The pentagonal dodecahedron coloured so as to preserve only central symmetry. The solid has a plane of symmetry with respect to each of the 3 colours considered individually.



The icosahedron coloured so as to preserve only central symmetry. The solid has a plane of symmetry with respect to green and yellow considered individually but not blue.

The final part of the workshop

Here we have a chance to review what we have done so far, in particular to cement the connection between reflections and rotations.

The order of the *full* symmetry group of the cube: a first look

Begin the review with words to the following effect:

"When studying rotations, we treated the cube as a solid block of wood. When studying reflections, we treated it as something you could turn inside out.

"We found there were 24 ways of turning a cube so that you wouldn't know whether it had been turned or not. We called this number the order of the rotation symmetry group of the cube.

"If we allow reflections as well as rotations when considering how many ways a cube can be changed to look like itself, we must double the order since, for every way of fitting the cube in its hole, we can either keep it as it is or turn it inside out. The order of the so-called *full* symmetry group of the cube is therefore 2 x 24 = 48.

The order of the full symmetry group of the cube: a second look

At this point we review experiments **E13**, **E10** respectively, reinterpreting a rotation as the product of two reflections.

Acetates A17, A18

Show A18 again. Point out that, to get from A to C, we can reflect in two perpendicular mirrors *or* perform a half-turn about their line of intersection. Discuss ways of getting from A to G.
[Rotation, reflection = reflection-reflection, reflection;
Reflection, rotation = reflection, reflection-reflection;
Reflection, reflection, reflection.]

Show A17 again. Ask the children to look again at the move from frame 4 to frame 5. Remind them that frame 4 shows a reflection, frame 5 a half-turn about an axis perpendicular to that plane. Point out that, if you think of the half-turns as two successive reflections, as in A18, you can get from the left of frame 4 to the right of frame 5 – i.e. back to the original – by performing 3 reflections in mutually perpendicular mirrors.

Demountable kaleidoscope, x 15

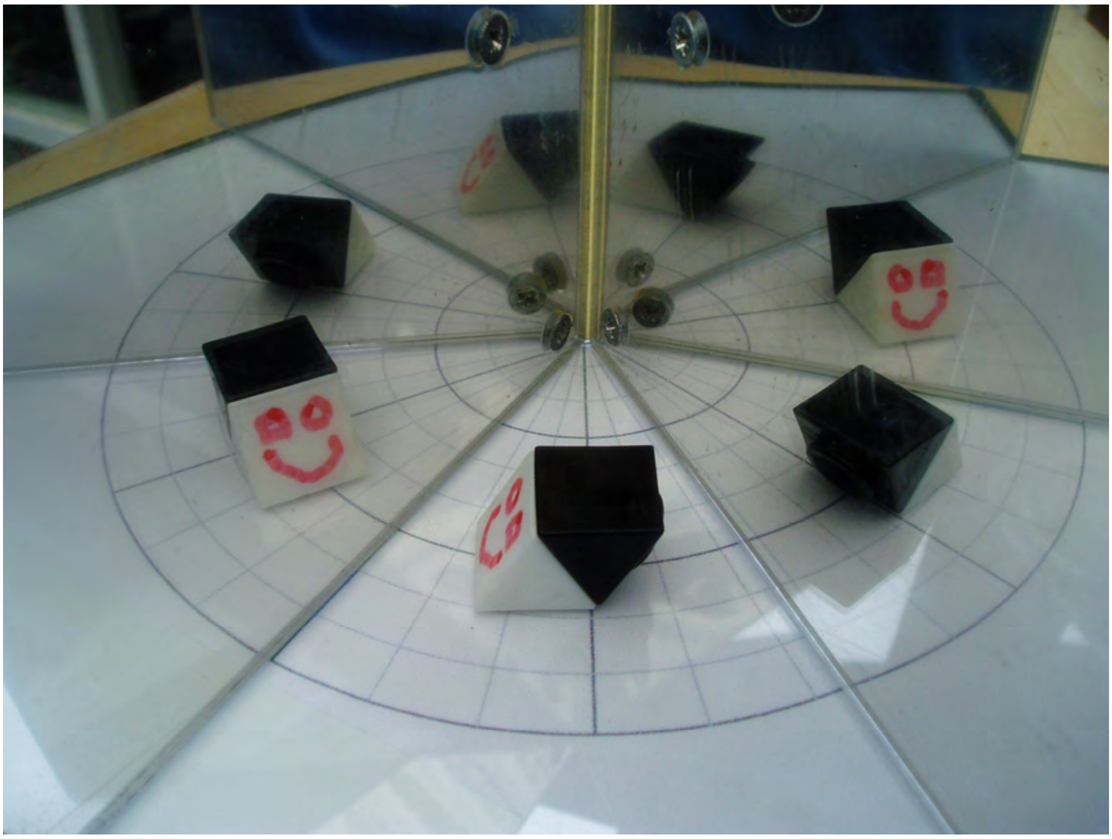
Intelocking cubes, dry-wipe pens as needed

E15 *Pupil experiment*

Give the children a little time to explore the kaleidoscope.

the children should build or draw something asymmetric between the mirrors, and alter the dihedral angle.

Ask them to comment on the *first* reflections in the two mirrors.



[The two images are the opposite way round to the object, and therefore the *same* way round. The angle between them is twice the angle between the mirrors.]

Point out that what we noticed in the previous experiments: reflection in two perpendicular mirrors gives a half-turn, is just a special case of the general statement: reflection in two mirrors set at a certain angle is equivalent to a rotation of twice this angle about their line of intersection.

Tell the children the significance of this: we can perform *all* symmetry operations by means of reflections alone.

Demountable kaleidoscope, extra mirrors added as described, x 15

20 mm wood cubes, whole or dissected, as needed

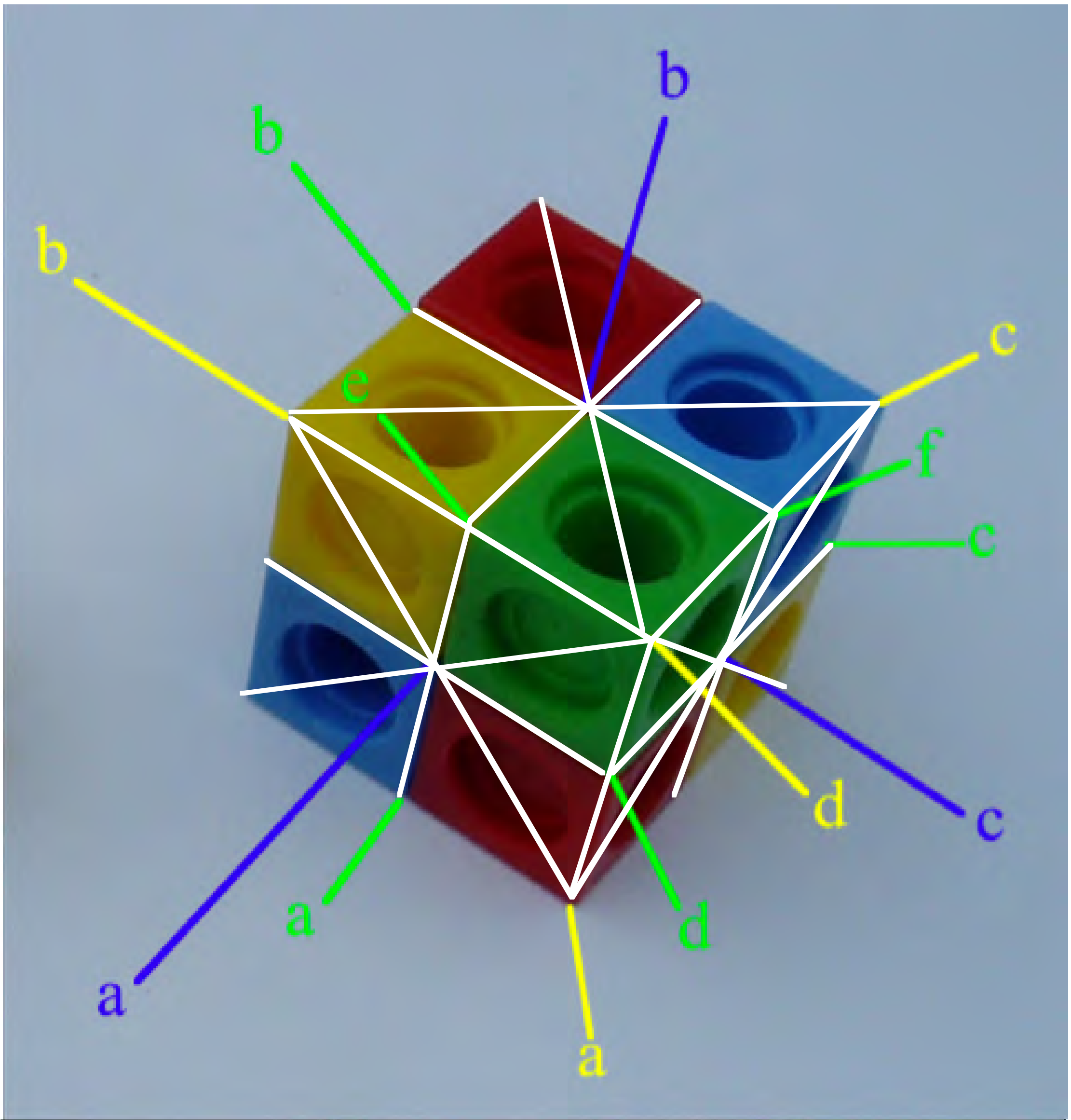
E16$_1$ *Pupil experiment*

Put A19 on the OHP (to remain there throughout **E16**). Remind the children of ff. experiments:

> **E5**, where they found the different ways they could turn a wooden cube and locate it in their hand;
> **E6**, where they did the same but using their 4-colour cubes and special boards;
> **E3**, where they located the rotation axes and mirror planes of a perspex cube filled with water.

Point out that A19 is exactly the same picture as the one they used to name their rotations (A10), but with the cube's symmetry planes marked in. Remind them that these are of two types:

> one contains only 'blue' and 'green' axes;
> the other contains 'blue', 'yellow' and 'green' axes.



Make all turns clockwise looking into the cube.

a, a^2, a^3, a^4 = 1/4, 2/4, 3/4, 4/4 of a turn about that axis, so a^4 brings you back to the start and is therefore just the identity operation, e.

a, a^2, a^3 = 1/3, 2/3, 3/3 of a turn, so $a^3 = e$.

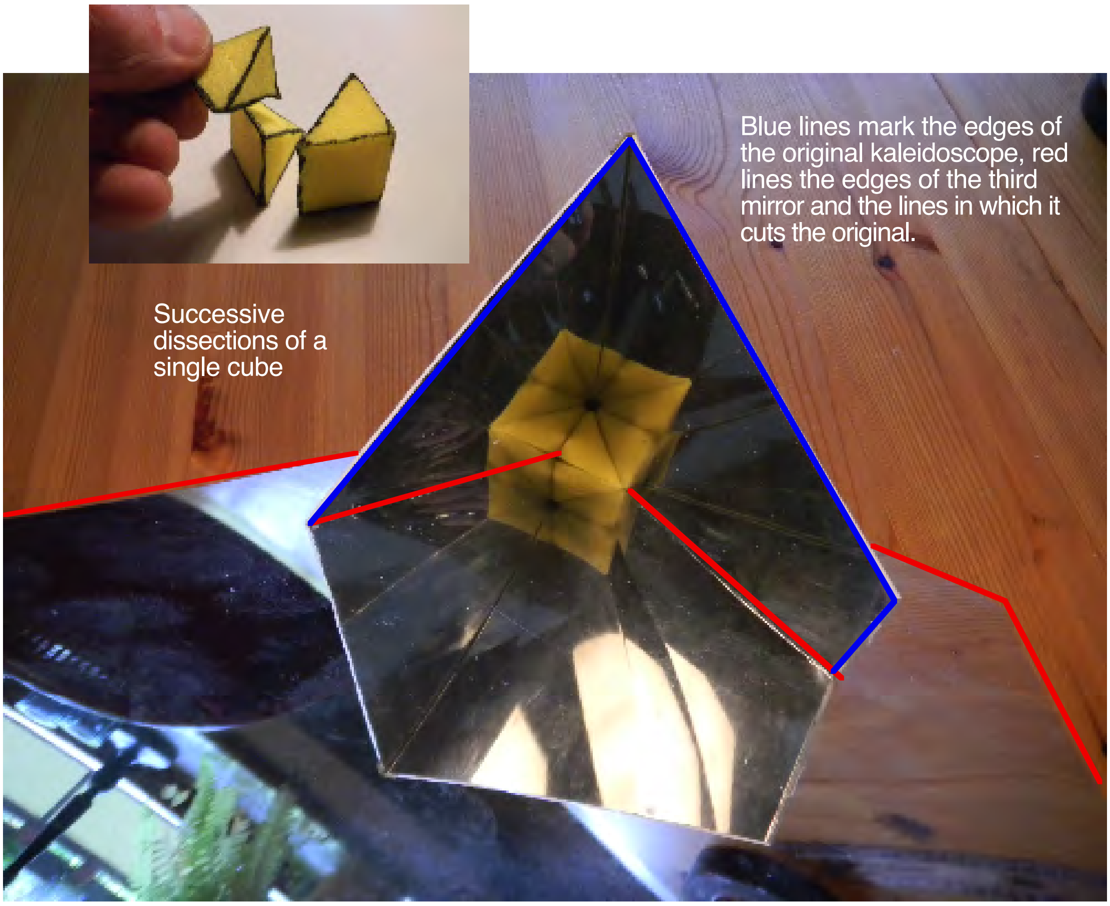
a, a^2 = 1/2, 2/2 of a turn, so $a^2 = e$.

Tell the children that their experiment will proceed in 6 steps.
Ask them to keep looking back at the picture and think how each step relates to the white lines marked there.

The aim is to realise the image of a 2 x 2 x 2 cube.
1. Each pair simply arranges accordingly 8 cubes on the table.
2. They then set their kaleidoscope at a straight angle and arrange 4 cubes so that, combined with the reflection, they have their larger cube.
3. They now set it at a right angle, fitting a stack of 2 cubes to realise the same overall image.
4. Now give them a mirror to set horizontally beneath their kaleidoscope, (thus converting their dihedral kaleidoscope into a polyhedral one). All they now need do is set 1 cube in the corner.
5. The next step is to reduce the dihedral angle to $45°$. Give each group a half-cube, sliced by a symmetry plane containing a pair of opposite edges.
6. Finally, swap the horizontal mirror for one designed to fit the $45°$ dihedral kaleidoscope at $45°$ to the horizontal. What you now give them is the previous piece sliced by a further, corresponding cut to produce an *orthoscheme*. This is a tetrahedron with the same base and height as the preceding prism and therefore with 1/3 of its volume.

Ask the children to calculate how many wooden pieces of the last kind make up a complete 2 x 2 x 2 cube. [Proceeding through the steps, we have: (**2.**) $1/2$ x (**3.**) $1/2$ x (**4.**) $1/2$ x (**5.**) $1/2$ x (**6.**) $1/3 = 1/48$.]

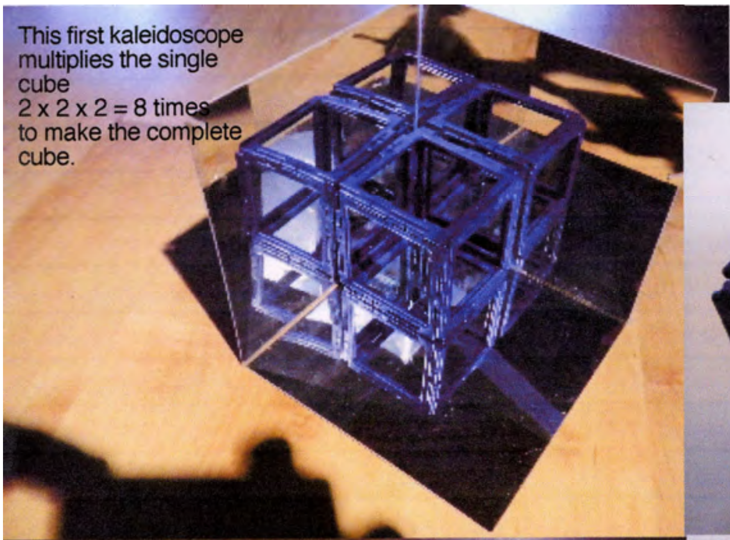
Point out to the children that we could have taken any one of 48 such pieces and fitted it into a kaleidoscope like ours to create the whole. We can therefore say that the order of the full symmetry group of the cube is 48.



Kaleidoscope as for **E16**$_1$**,** Polydron Frameworks: squares: 8 blue, right-angled isosceles triangles: 10 blue, 2 yellow, x 15

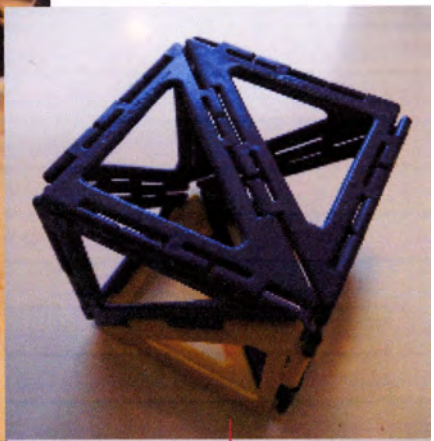
E16$_2$ (variant on **E16**$_1$)

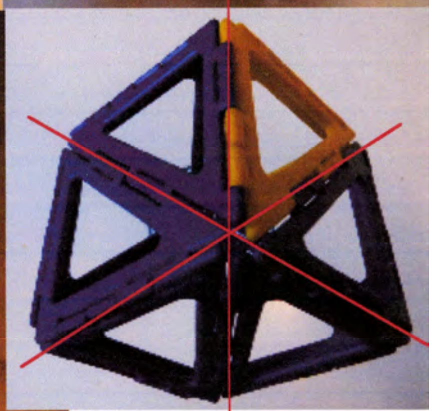
We pass straight to step **4** in **E16**$_1$, using Polydron Frameworks in place of wood for our single cube and its dissections. However, we omit step **5** and go straight to step **6**.



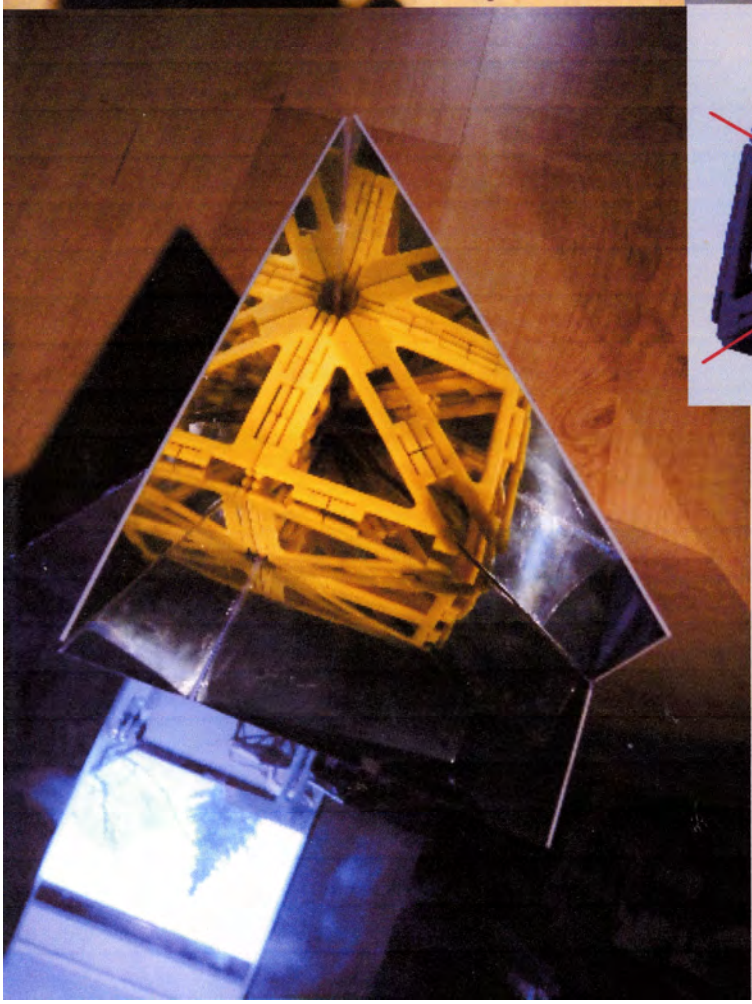
This first kaleidoscope multiplies the single cube
2 x 2 x 2 = 8 times
to make the complete cube.

We cut from the single cube the yellow part (an 'orthoscheme').

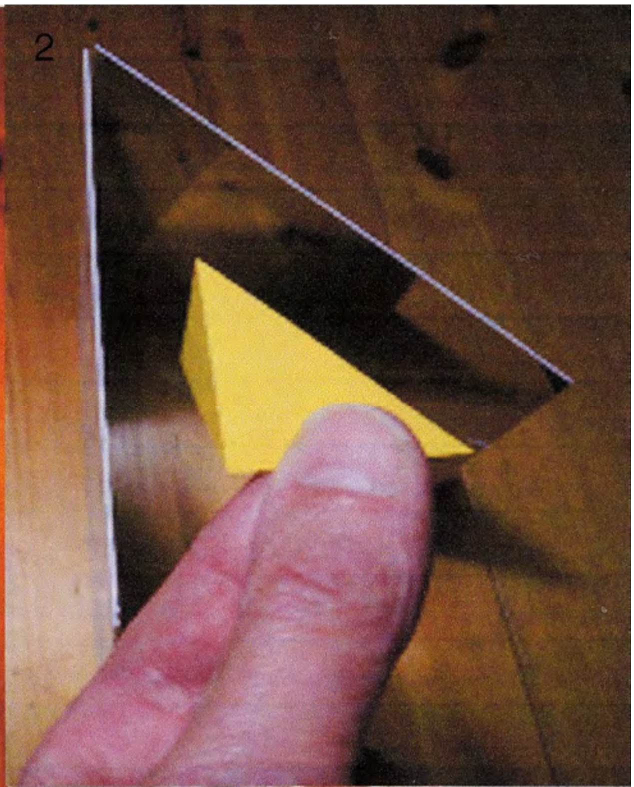




Looking down a 3-axis, we see from the planes of symmetry that this occupies 1/6 of a single cube.



This second kaleidoscope must therefore multiply the yellow part
6 x 8 = 48 times
to make the complete cube.



1 A dihedral kaleidoscope is set at 45⁰. The slits in the mirror lying flat on the table engage with the two arrowed slots. When slid into place, this mirror is perpendicular to the left-hand flap of the dihedral kaleidoscope and inclined at 45⁰ to the table. You also see the net of the orthoscheme in yellow card.

2 The folded orthoscheme in the orientation it adopts when the inclined mirror is in place.

3 The result.

THE CUBE AND ITS SYMMETRIES (short version)	**minutes allowed**	**pp. in original** text	**experiment**
DEFINITION			
In how many ways can you move something so that it looks as if it hasn't moved at all?		1	
In other words, in how many ways can something fit its own space?			
shut eyes & box	**(5)**		
This is the number of *symmetries* it has.			
The 2 kinds of symmetry: *rotational* and *reflexive*:			
Centres and mirror *lines* shown in 2-D: **acetates**	**5**	2 – 8	**E1**
Note: 2 90 degree reflections = 1 180 degree rotation			
Axes and mirror *planes* shown in 3-D:			
water cube	**10**	12 – 13	**E3**
destroying the cube's symmetry	**10**	14	**E4**
ROTATIONS			
Axes and all possible rotations:	**5**	14 - 15	**E5**
beechwood cube (giving order 24),			
demonstration cube to recap.			
Link:			
space diagonal model (4-colour cube)	**5**	16	**E6** (cube only)

The rotations correspond to all the ways you can *permute* the space diagonals.
We can have any of 4 in the first position combined with any of the remaining 3 in the second position combined with either of the remaining 2 in the third position, and that fixes the fourth. The possibilities multiply up to give … ?

REFLECTIONS			
By swapping front and back:			
L & R 8-colour cubes + Mira	**5**	25 – 30	**E8**
By turning inside out:			
Polydron L glove to right	**10**	30 – 31	**E9**

Does an object have to have a mirror plane to be *superimposable*?

Central symmetry:			
Polydron 4-colour cube turned inside out	**10**	32 – 33	**E10**
demonstration models of distorted cuboids retaining central symmetry		34 – 35	

dihedral kaleidoscope + upright fist to return to fact: 2 reflections at angle A = rotation of 2A.

We can perform all rotations by means of reflections alone.	**5**	41	**E15**

Mirror cube + fist to show why centrally symmetric objects are superimposable (and answer to above question: No).

build-up of polyhedral kaleidoscope to Coxeter case,	**10**	44 – 45	**E16**

when *full* order of symmetry is 2 x 2 x 2 x 2 x 3 = 48.
Link: do rotations but swap front and back each time: 24 x 2 also = 48.

P.S. 22.12.11